PHYS490 Independent Study II: Scalar Tensor Theories

Fethi M. Ramazanoğlu[1] and Ekrem S. Demirboğa[1]

[1] *Department of Physics, Koç University, Rumelifeneri Yolu, 34450 Sariyer, Istanbul, Turkey*

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In this study we investigated a class of alternative gravity theories called scalar-tensor theories and a phenomenon called spontenous scalarization that occurs in certain scalar-tensor theories. More specifically, we derived field equation and the corresponding Tolman–Oppenheimer–Volkoff (TOV) equations equations for the massive scalar-tensor theory and compared it with the General Relativity (GR). In addition, necessary tools for numerical analysis such as C and C++ are studied.

I. INTRODUCTION

In General Relativity (GR), the metric g is the only dynamical field and there are no arbitrary functions nor parameters other than the Newtonian coupling constant G. The equation of motions can be obtained (see Appendix B) from the following action

$$S = \int d^4x\sqrt{-g}(R + \mathcal{L}_M)$$

which are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

In scalar Tensor Theories, in addition to the metric field g we have one or several scalar field φ coupled to curvature scalar R and the action takes the form

$$S = \int d^4x\sqrt{-g}\bigg(f(\varphi)R - \tfrac{1}{2}h(\varphi)g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - V(\varphi) + \mathcal{L}_M(g_{\mu\nu}, \psi_M)\bigg)$$

where the functions f,h and V defines the theory and matter langrangian depends only on the metric and matter fields. One of the earliest example is the Brans-Dicke theory and corresponds to ([1])

$$f(\varphi) = \frac{\varphi}{16\pi}, \qquad h(\varphi) = \frac{\omega}{8\pi\varphi}$$

A common approach to scalar-tensor theories to perform a conformal transformation to the action by defining a conformal metric

$$\tilde{g} = \frac{1}{16\pi f(\varphi)}g_{\mu\nu}$$

where we used ˜ to denote the old metric (*Jordan Frame*). And we end up with following action

$$S = (16\pi G)^{-1}\int d^4x\sqrt{-g}\left[R - \tfrac{3}{2}g^{\rho\sigma}f^{-2}\left(\frac{df}{d\varphi}\right)^2(\partial_\rho\varphi)(\partial_\sigma\varphi)\right]$$

This frame called Einstein frame and will be shown without ˜. We will be working on this frame throughout this paper.

II. SPONTANEOUS SCALARIZATION

A. Massless Scalar Field

The action for the scalar tensor theories for one arbitrary coupling function $A(\varphi)$ can be written as in [2]

$$S = \frac{1}{16\pi G}(S_H + S_\varphi) + S_M[\psi_M, A^2(\varphi)g_{\mu\nu}] \tag{1}$$

where g being the Einstein metric and related to Jordan metric as

$$\tilde{g}_{\mu\nu} = A^2 g_{\mu\nu} \tag{2}$$

As noted earlier, even though the metric g is the one we measure in experiments it is easier the formulate the theory in Einstein frame.

From equation (1) we end up with the following field equations (see Appendix C 1)

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{3a}$$

$$\Box\varphi = -4\pi G\alpha(\varphi)T \tag{3b}$$

where $T^{\mu\nu} = 2\sqrt{g}\delta S_M/\delta g_{\mu\nu}$ is the stress energy tensor and $\alpha = \partial\ln A/\partial\varphi$ is the coupling strength between scalar field between scalar and matter fields.

Now, consider the the static case where $A(\varphi) = e^{\beta\varphi^2/2}$ and β is a negative constant. If we neglect the gravitational field and nonlinear terms in φ we can write the equation 3b as

$$\Box\varphi = -4\pi\tilde{T}\beta\varphi$$

where we write in terms of the physical energy-momentum tensor (to see the transformation see Appendix A) and take $G = 1$ for simplicity. Assuming nonrelativistic matter, i.e. $\rho \gg p$, therefore, $\tilde{T} \approx -\rho$, then we have

$$\Delta\varphi = 4\pi\rho\beta\varphi \tag{4}$$

Then, setting $k^2 = \beta\rho$ we have the following solutions ([2], [3])

$$\mathcal{A}\frac{sin(kr)}{r}, \qquad r \leq R \tag{5a}$$

$$\frac{\mathcal{M}}{r} + \varphi_0, \qquad r \geq R \tag{5b}$$

where R is the radius of the star and $\mathcal{A}$ and $\mathcal{M}$ is constant. And the continuity condition of φ and φ' at the R gives

$$\mathcal{A} = \frac{\varphi_0}{kcos(kR)}, \tag{6a}$$

$$\mathcal{M} = \varphi_0[k^{-1}tan(kR) - R]. \tag{6b}$$

From this it can be seen that as $KR \to \pi/2$, φ and $\mathcal{M}$ greatly increases, i.e. Spontaneous Scalarization occurs ([3]).

B. Massive Scalar Field

For the massive scalar field we still can write the action in the form of equation 1 with the addition of the mass term to S_φ as ([4])

$$S = \frac{1}{16\pi}\int d^4x\sqrt{g}[R - 2g_{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2 + \mathcal{L}_M] \tag{7}$$

which gives the equation of motions as (see Appendix C 2)

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2 g_{\mu\nu} + 8\pi(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{8a}$$

$$\Box\varphi = -4\pi\alpha(\varphi)T + m^2\phi \tag{8b}$$

Now consider the same case with $A = e^{\beta\varphi^2}$ where β is a negative constant. We again write equation using the jordan frame energy-momentum tensor as before (Appendix A).

$$\Box\varphi = -4\pi\beta e^{2\beta\varphi^2}\varphi\tilde{T} + m^2\phi$$

Note that, $\varphi = 0$ is the GR solution. Perturbating around this solution and expanding to linear order and with the assumption of nonrelativistic matter we have

$$\Delta\varphi = \left(4\pi\beta\tilde{\rho} + m^2\right)\varphi \tag{9}$$

which is very similar to what we had in equation 4. Therefore, with a similar approach we can argue that Spontaneous Scalarization occurs with similar solutions. Or following the discussion in [4], the first term on the right in equation 9 is effectively a negative mass-squared term. Therefore, equation suffers a tachyon instability. This equation has the solutions of the form for $r < R$

$$\varphi \backsim C_1\frac{e^{ikr}}{r} + C_2\frac{e^{ikr}}{kr} \tag{10}$$

where we defined $k = \frac{2\pi}{\lambda} = \sqrt{4\pi\beta\tilde{\rho} + m^2}$. Therefore we have an instability if $m^2 < 4\pi\beta|\rho| \implies \lambda_\varphi > \lambda_{eff}$ where

$$\lambda_{eff} = \sqrt{\frac{\pi}{|\beta|\rho}} \tag{11}$$

Hence, all fourier modes with wavelength $\lambda > \lambda_{eff}/\sqrt{1-(\lambda_{eff}/\lambda_\varphi)^2}$ with $\lambda_{eff} < \lambda_\varphi$ will initially experience an exponential growth. However the $e^{2\beta\varphi^2}$ term will initial take over at order of $1/\sqrt{|\beta|}$.

For a star we can write approximalty $\tilde{\rho} \approx M/R^3$, then,

$$\lambda_{eff,star} \backsim \frac{R}{\sqrt{C|\beta|}} \tag{12}$$

where $C = 2M/R$ is the compactness of the star. Also, for spontenous scalarization to occur, we must have $\lambda_{eff} < \lambda_\varphi$ and shortest unstable mode must fit in the star, i.e. $\lambda_{eff} < R$, therefore, for given β only the stars that have

$$C \gtrsim 1/|\beta \tag{13}$$

can scalarize.

One other way the look at the spontenious scalarization scenario is the write the field equation for φ as in [5]

$$\Box\varphi - \partial_\varphi V_{eff}(\varphi) = 0 \tag{14}$$

where we define the effective potential $V_{eff}(\varphi)$ and make the same assumptions as before but we choose $A^2(\varphi) = 1 - \epsilon + \epsilon e^{\phi^2/2M^2}$ just to be able to integrate V_{eff} easily. But as shown before the same scenario aplies for other functions too. Then we have the following effective potential

$$V_{eff}(\varphi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}(1 - \epsilon + \epsilon e^{\phi^2/2M^2})\tilde{\rho} \tag{15}$$

For uniform $\tilde{\rho}$ system is static and scalar field φ takes a constant value that minimizes the effective potential Taylor expanding the effective potential around $\varphi = 0$ gives

$$V_{eff}(\varphi) = \frac{1}{4}\tilde{\rho} + \frac{1}{2}\left(m^2 - \frac{\epsilon\tilde{\rho}}{2M^2}\right)\varphi^2 + \mathcal{O}(\varphi^4) \tag{16}$$

It can be seen that $\varphi = 0$ is stable for $\tilde{\rho} < \rho_{PT} \equiv 2m^2M^2/\epsilon$ and unstable otherwise . And it takes the form ([5])

$$\frac{\varphi^2}{2M^2} = \ln\left[\frac{2\epsilon}{1-\epsilon}\left(\sqrt{1 + \frac{4\epsilon(\rho_{PT})}{\tilde{\rho}(1-\epsilon)^2}} - 1\right)^{-1}\right]$$

Therefore, φ proportional to $\tilde{\rho}$ logarithmically. Therefore, if $\tilde{\rho}$ doesn't takes extremely high values φ is order of M, hence, scalar field scalarized when $\tilde{\rho} < \rho_{PT}$.

III. TOV EQUATIONS IN SCALAR-TENSOR THEORIES

Tolman–Oppenheimer–Volkoff (TOV) equations are the hydrostatic equilibrium equations for spherically symmetric bodies, in other words, stars. In GR they take the form

$$\begin{aligned} m' &= 4\pi r^2 \rho \\ \nu' &= 2\frac{m + 4\pi r^3 p}{r(r-2m)} \\ p' &= -\frac{m + 4\pi r^3 p}{r(r-2m)}(\rho + p) = -\frac{1}{2}(p+\rho)\nu' \end{aligned}$$

with the metric

$$ds^2 = -e^{\nu(r)}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

In the following chapters, we will derive the corresponding TOV equations for scalar-tensor theories.

A. TOV Equations in Massless Scalar-Tensor Theories

The spherically symmetric, static metric, generated by an isolated, non-rotating neutron star can be written in Einstein Frame as[2]

$$ds^2 = -e^{\nu(r)}dt^2 + \left(1 - \frac{2\mu(r)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

And the stress energy tensor in Jordan frame $\tilde{T}^{\mu\nu}$ takes the perfect fluid form as

$$\tilde{T}^{\mu\nu} = (\tilde{\epsilon} + \tilde{p})\tilde{u}^\mu \tilde{u}^\nu + \tilde{p}\tilde{g}^{\mu\nu} \tag{17}$$

or in matrix form

$$\tilde{T}^{\mu\nu} = \begin{pmatrix} e^{-\nu}\tilde{\epsilon}A^{-2} & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2\mu}{r}\right)\tilde{p}A^{-2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2}\tilde{p}A^{-2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 sin^2\theta \tilde{p}}A^{-2} \end{pmatrix}$$

Calculating the trace we get

$$\tilde{T}^\mu \nu \tilde{g}_{\mu\nu} = \tilde{T} = -\tilde{\epsilon} + 3\tilde{p} \tag{18}$$

Similarly, in Einstein frame, we have $T^{\mu\nu} = \tilde{T}^{\mu\nu}A^6$ from which we can get

$$T = A^4(-\tilde{\epsilon} + 3\tilde{p}) = A^4\tilde{T} \tag{19}$$

We will also write the energy momentum tensor in Einstein frame for future references

$$T^{\mu\nu} = \begin{pmatrix} e^{\nu}\tilde{\epsilon}A^4 & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2\mu}{r}\right)^{-1}\tilde{p}A^4 & 0 & 0 \\ 0 & 0 & r^2\tilde{p}A^4 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta\tilde{p}A^4 \end{pmatrix}$$

Now, with the help of Mathematica we find (tt) and (rr) components of the corresponding Einstein tensor to be

$$\begin{aligned} G_{tt} &= \frac{2e^\nu \mu'}{r^2} \\ G_{rr} &= \frac{\nu'}{r} - \frac{2\mu}{r^2(r-2\mu)} \end{aligned} \tag{20}$$

where prime denotes derivative with respect to r (i.e. d/dr).

1. $\mu(r)$ Equation

We start by writing the (tt) component of the equation (C3). Note that since we are dealing with spherically symmetric and static system, there is only r dependence in φ.

$$\begin{aligned} G_{tt} =& - g_{tt}g^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi + 2\partial_t\varphi\partial_t\varphi + 8\pi G T_{tt} \\ \frac{2e^\nu\mu'}{r^2} =& e^\nu\left(1 - \frac{2\mu}{r}\right)\psi^2 + 8\pi G A^4 e^\nu \tilde{\epsilon} \\ \mu' =& \frac{1}{2}r(r-2\mu)\psi^2 + 4\pi G r^2 A^4 \tilde{\epsilon} \end{aligned} \tag{21}$$

where we defined $\psi = \varphi'$.

2. $\nu(r)$ Equation

Similarly for (rr) component writing the equation we have:

$$G_{rr} = - g_{rr}g^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi + 2\partial_r\varphi\partial_r\varphi + 8\pi G T_{rr}\frac{\nu'}{r} - \frac{2\mu}{r^2(r-2\mu)} \tag{22}$$

$$\begin{aligned} &= -\left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{2\mu}{r}\right)^{-1}\psi^2 + 2\psi^2 + 8\pi G A^4 \tilde{p}\left(1 - \frac{2\mu}{r}\right)^{-1} \\ \nu' =& 8\pi G A^4 \frac{r^2}{r-2\mu} + \psi^2 r + \frac{2\mu}{r^2(r-2\mu)} \end{aligned} \tag{23}$$

Notice that the second term in right hand side did not vanish.

3. $\psi(r)$ Equation

Now, we write down the equation (C9), recalling (19)

$$\begin{aligned} \Box\varphi =& - 4\pi G\alpha(\varphi)T \\ g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi =& - 4\pi G\alpha A^4(-\tilde{\epsilon} + 3\tilde{p}) \end{aligned}$$

calculating the left hand side gives (writing only nonzero Christoffel symbols)

$$\begin{aligned} g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi =& g^{\mu\nu}\nabla_\mu\partial_\nu\varphi \\ =& g^{\mu\nu}\partial_\mu\partial_\nu\varphi - \Gamma^k_{\mu\nu}\partial_k\varphi \\ =& \underbrace{g^{rr}\partial_r^2\varphi}_{\text{I}} - \underbrace{g^{rr}\Gamma^r_{rr}\partial_r}_{\text{II}} - \underbrace{g^{\theta\theta}\Gamma^r_{\theta\theta}\partial_r}_{\text{III}} - \underbrace{g^{tt}\Gamma^r_{tt}\partial_r}_{\text{IV}} - \underbrace{g^{\phi\phi}\Gamma^r_{\phi\phi}\partial_r}_{\text{V}} \end{aligned}$$

Calculating each term one by one gives the following results

$$\begin{aligned}
I \Longrightarrow & \left(\frac{r-2\mu}{r}\right)\psi' \\
II \Longrightarrow & -\left(\frac{\mu - r\mu'}{r^2}\right)\psi \\
III \Longrightarrow & -\frac{r-2\mu}{r^2}\psi \\
IV \Longrightarrow & \frac{-\nu'(r-2\mu)}{2r}\psi \\
V \Longrightarrow & -\frac{r-2\mu}{r^2}
\end{aligned} \tag{24}$$

Putting all these together yields

$$\begin{aligned}
g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = & \left(\frac{r-2\mu}{r}\right)\psi' + \left(\frac{\mu - r\mu'}{r^2}\right)\psi + \frac{r-2\mu}{r^2}\psi \\
& + \frac{\nu'(r-2\mu)}{2r}\psi + \frac{r-2\mu}{r^2}\psi
\end{aligned} \tag{25}$$

In order to find an expression for ψ we substitute μ' and ν' from equations (21) and (22). After getting everything together and rearranging yields the following

$$\begin{aligned}
= & \left(\frac{r-2\mu}{r}\right)\psi' + \frac{\mu}{r^2}\psi - 4\pi G r A^4 \tilde{\epsilon}\psi - \frac{1}{2}(r-2\mu)\psi^3 \\
& + \frac{r-2\mu}{r^2}\psi + \frac{\mu}{r^2}\psi + 4\pi G r A^4 \tilde{p}\psi \\
& + \frac{1}{2}(r-2\mu)\psi^3 + \frac{r-2\mu}{r^2} \\
= & 4\pi G\alpha A^4(\tilde{\epsilon} - 3\tilde{p})
\end{aligned}$$

Rearranging and Simplifying this yields

$$\begin{aligned}
\psi' = & \left(4\pi G\alpha A^4(\tilde{\epsilon} - 3\tilde{p}) + 4\pi G r A^4(\tilde{\epsilon} - \tilde{p})\psi - \frac{2(r-\mu)}{r^2}\psi\right)\left(\frac{r}{r-2\mu}\right) \\
= & 4\pi G\frac{rA^4}{r-2\mu}\left(\alpha(\tilde{\epsilon} - 3\tilde{p}) + r\psi(\tilde{\epsilon} - \tilde{p})\right) - \frac{2(r-\mu)}{r(r-2\mu)}\psi
\end{aligned} \tag{26}$$

Which is the equation for ψ.

4. $\tilde{p}(r)$ Equation

Here we will use stress energy balance equation instead of $(\theta\theta)$ equation. Therefore we have

$$\begin{aligned}
\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = & 0 \\
= & \partial_\mu \tilde{T}^{\mu\nu} + \tilde{\Gamma}^\mu_{\mu\lambda}\tilde{T}^{\lambda\nu} + \tilde{\Gamma}^\nu_{\mu\lambda}\tilde{T}^{\mu\lambda}
\end{aligned} \tag{27}$$

where every component except for the r component trivially vanishes. Hence, for $\nu = r$, writing only the nonvanishing terms we have

$$\begin{aligned}
\tilde{\Gamma}_\mu \tilde{T}^{\mu r} = & \partial_r \tilde{T}^{rr} + \tilde{\Gamma}^\mu_{\mu r}\tilde{T}^{rr} + \tilde{\Gamma}^r_{\lambda\lambda}\tilde{T}^{\lambda\lambda} \\
= & \partial_r \tilde{T}^{rr} + \left(\tilde{\Gamma}^t_{tr} + \tilde{\Gamma}^r_{rr} + \tilde{\Gamma}^\theta_{\theta r} + \tilde{\Gamma}^\varphi_{\varphi r} + 2\tilde{\Gamma}^r_{rr}\right)\tilde{T}^{rr} + \tilde{\Gamma}^r_{tt}\tilde{T}^{tt} + \tilde{\Gamma}^r_{\theta\theta}\tilde{T}^{\theta\theta} + \tilde{\Gamma}^r_{\varphi\varphi}\tilde{T}^{\varphi\varphi} \\
= & -\frac{2\mu'}{r} + \frac{2\mu}{r^2}\tilde{p}A^{-2} + \left(\frac{r-2\mu}{r}\right)A^{-2}\tilde{p}' + \left(\frac{r-2\mu}{r}\right)\tilde{p} - 2A^{-3}A' \\
& + \left[\left(\frac{A'}{A} + \frac{\nu'}{2}\right) + 2\left(\frac{A'}{A} - \frac{\mu - r\mu'}{r(r-2\mu)}\right) + \left(\frac{A'}{A} + \frac{1}{r}\right) + \left(\frac{A'}{A} + \frac{1}{r}\right)\right]\left(\frac{r-2\mu}{r}\tilde{p}A^{-2}\right) \\
& + \frac{(r-2\mu)(2A' + A\nu')e^\nu e^{-\nu}}{2r}\tilde{\epsilon}A^{-3} - \frac{(r-2\mu)(A + rA')}{r^2}\tilde{p}A^{-3} - \frac{-sin^2\theta(r-2\mu)(A+rA')}{r^2 sin^2\theta}\tilde{p}A^{-3}
\end{aligned} \tag{28}$$

where $A' = dA/dr$.Note that we can write this as

$$\frac{dA}{dr} = \frac{\partial A}{\partial\varphi}\frac{d\phi}{dr} = A\alpha\psi$$

Putting this into the equation we can write for $\tilde{p}'$ as

$$\begin{aligned}
\tilde{p}' = & \frac{2\mu'}{r-2\mu} - \frac{2\mu}{r(r-2\mu)}\tilde{p} + 2\tilde{p}\alpha\psi - 3\tilde{p}\alpha\psi - 2\tilde{p}\alpha\psi - \frac{\nu'}{2}\tilde{p} + \frac{2\mu}{r(r-2\mu)}\tilde{p} \\
= & -(\tilde{\epsilon} + \tilde{p})\left(\frac{\nu'}{2} + \alpha\psi\right)
\end{aligned} \tag{29}$$

Substituting ν' and μ' we get the equation for p

$$\tilde{p}' = -(\tilde{p} + \tilde{\epsilon})\left[4\pi G\frac{r^2A^4\tilde{p}}{r-2\mu} + \frac{r}{2}\psi^2 - \frac{\mu}{r(r-2\mu)} + \alpha\psi\right] \tag{30}$$

Therefore togather with the equations 21, 22, 26 and 30 we have our TOV equations.

B. TOV Equations in Massive Scalar-Tensor Theories

We have the field equations as (see Appendix C 2)

$$\begin{aligned} R_{\mu\nu} =& 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + 2\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2 g_{\mu\nu} \\ \Box\varphi =& -4\pi\alpha(\varphi)T + m^2\varphi \end{aligned} \tag{31}$$

and the same metric as before. Using a very similar route we see that tt equation read

$$\begin{aligned} G_{tt} =& -g_{tt}g^{\rho\sigma}\partial_\sigma\varphi\partial_\rho\varphi + 8\pi T_{tt} - m^2\varphi^2 g_{tt} \\ \frac{2e^{\nu}\mu'}{r^2} =& e^{\nu}\left(\frac{r-2\mu}{r}\right)\psi + 8\pi A^4 e^{\nu}\tilde{\rho} + e^{\nu}m^2\varphi^2 \end{aligned}$$

From this we can write

$$\mu' = \frac{r-2\mu}{2}\psi + 4\pi A^4 r^2\tilde{\rho} + \frac{1}{2}m^2r^2\varphi^2 \tag{32}$$

where we again defined $\varphi' = \psi$. Now, Similarly we write the rr equation

$$G_{rr} = -g_{rr}g^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi + 2(\partial_r\varphi)^2 + 8\pi T_{rr} - m^2\varphi^2 g_{rr}$$

which reduces to

$$\begin{aligned} \frac{\nu'}{r} - \frac{2\mu}{r^2(r-2\mu)} =& -\psi^2 + 2\psi^2 + 8\pi A^4\tilde{p}\left(\frac{r}{r-2\mu}\right) \\ & - m^2\varphi^2\left(\frac{r}{r-2\mu}\right) \end{aligned} \tag{33}$$

after some simplification we write the equation for ν as

$$\nu' = r\psi^2 + \frac{1}{den}\left[8\pi A^4\tilde{p}r^3 - m^2\varphi^2r^3 + 2\mu\right] \tag{34}$$

And again from the field equation for φ we can write

$$g^{\mu\nu}\partial_\mu\partial_\nu\varphi = -4\pi\alpha A^4(-\rho \tilde{+} 3\tilde{p}) + m^2\phi$$

We calculate the left hand side just as before

$$\begin{aligned} g^{\mu\nu}\partial_\mu\partial_\nu\varphi =& \left(\frac{r-2\mu}{r}\right) + \frac{\mu}{r} - 4\pi A^4 r\tilde{\rho}\psi - \frac{m^2r\varphi\psi}{2} \\ & + \frac{2\psi}{r} - \frac{4\mu}{r^2}\psi + 4\pi A^4\tilde{p}r\psi - \frac{1}{2}m^2\varphi^2 r\psi \\ & + \frac{2\mu}{r^2} + \frac{r-2\mu}{r^2}\psi \end{aligned}$$

By simplifying this we can write

$$\begin{aligned} (r-2\mu)\psi' =& 4\pi r A^4[\alpha(\rho \tilde{-} 3\tilde{p}) + r\psi(\tilde{\rho} - \tilde{p})] \\ & + m^2(r^2\varphi^2\psi + r\varphi) - 2\psi(1-\mu/r) \end{aligned} \tag{35}$$

Finally, we again look at the stress-energy balance equation which can be written using $\tilde{\nabla}g_{\mu\nu} = 0$ as

$$\tilde{\nabla}_\mu\tilde{T}^{\mu\nu} = \tilde{\nabla}_\mu[(\tilde{\rho} + \tilde{p})\tilde{u}^\mu\tilde{u}^\nu + \tilde{p}g^{\tilde{\mu}\nu}] = 0$$

Writing only the nonzero terms we have

$$\tilde{\nabla}_t(\tilde{\rho} + \tilde{p})\tilde{u}^t\tilde{u}^\nu + \tilde{p}'g^{rr} = 0$$

$$\begin{aligned} &\implies \Gamma^t_{tt}(\tilde{\rho} + \tilde{p})\tilde{u}^t\tilde{u}^t + \Gamma^r_{tt}(\tilde{\rho} + \tilde{p})\tilde{u}^t\tilde{u}^t + \tilde{p}'\left(\frac{r-2\mu}{r}\right)A^{-2} \\ &\implies (\tilde{\rho} + \tilde{p})\left(\frac{r-2\mu}{r}\right)\left(\frac{2A' + A\nu'}{2A}\right)A^{-2} + \tilde{p}\left(\frac{r-2\mu}{r}\right)A^{-2} = 0 \\ &\implies \tilde{p}' = -(\tilde{\rho} + \tilde{p})(\alpha\psi + \frac{\nu'}{2}) \end{aligned} \tag{36}$$

Therefore, the equations 32, 34, 35 and 36 are the TOV equations corresponding to massive scalar field.

IV. DISCUSSION

We discussed the physical mechanism behind the spontaneous scalarization from [3] [5] [4] and later derived the field equations and TOV equations for both massless and massive scalar theories. But the problem with the massless scalar-tensor theories is that recent observations ruled out most of its parameter space ([6] and it also effects the whole universe and therefore it is needed to add a mass term which is bounded as [4]

$$m \gg 10^{-16}eV \tag{37}$$

Again from [6] to allow neutron stars to scalarize but not the white dwarf, bounds the compactness in a way such that

$$3 \lesssim -\beta \lesssim 10^3 \tag{38}$$

Also mass term prevents scalarization if the following conditions is not satisfied [4]

$$m \lesssim 10^{-9}eV \tag{39}$$

Also, since φ decays exponentially in massive scalar field, it is expected that strong field of one star can induce a scalarization in another which can be detected by gravitational wave observations.[4]

V. CONCLUSION

In general, massive scalar-tensor theories are more likely to agree with observations as discussied in section IV which can cause these theories to deviate from GR for the neutron star structures while agreeing with the observations.

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Appendix A: Transformation of Energy Momentum Tensor Between Einstein and Jordan Frames

We have the metric in both frames related as

$$\tilde{g}_{\mu\nu} = A^2 g_{\mu\nu}$$

Therefore we have the relation for the determinant as

$$|\tilde{g}| = |g|A^8$$
$$\sqrt{-\tilde{g}} = A^4\sqrt{-g}$$

Using this we can show that

$$\begin{aligned}T^{\mu\nu} =& \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta \tilde{g}^{\mu\nu}}\frac{\partial \tilde{g}^{\mu\nu}}{\partial g^{\mu\nu}} \\ =& \frac{-2}{A^{-4}\sqrt{-\tilde{g}}}\frac{\delta S_m}{\delta \tilde{g}^7 \mu\nu}A^{-2} \\ =& \tilde{T}_{\mu\nu}A^2\end{aligned} \tag{A1}$$

Similarly we can show that[7]

$$T^{\mu\nu} = \tilde{T}^{\mu\nu}A^6 \tag{A2}$$

Appendix B: General Relativity

The Action in General relativity is given as

$$S = \frac{1}{16\pi G}S_H + S_M$$

where S_M is the matter piece of the action and S_H is the hilbert action and given as

$$S_H = \int d^4x\sqrt{-g}R$$
$$S_M = \int d^4x\sqrt{-g}\mathcal{L}_M$$

where R is the ricci scalar. Then, we demand this variation of the action with respect to the metric to be zero. But stationary points with respect to inverse metric is the same as the ones with respect to inverse metric. Therefore variation of the action with respect to inverse metric gives

$$\begin{aligned}\delta S =& \int d^4x \frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = 0 \\ =& \int d^4x\left(\frac{1}{16\pi G}\frac{\delta S_H}{\delta g^{\mu\nu}} + \frac{\delta S_M}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu 0} \\ =& \int d^4x\sqrt{-g}\left(\frac{1}{\sqrt{-g}}\frac{1}{16\pi G}\frac{\delta S_H}{\delta g^{\mu\nu}} + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu}\end{aligned}$$

From this the equation of motions can be written as

$$\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} = 8\pi G T_{\mu\nu}$$

where we defined

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{B1}$$

After some calculations left hand side of the equation can be found as

$$\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = G_{\mu\nu} \tag{B2}$$

Togather gives the Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{B3}$$

Or by taking the trace in an alternative version

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{B4}$$

Appendix C: Field Equations in Scalar Tensor Theories

1. Massless Scalar Field

We have the action for the scalar tensor theories for one arbitrary coupling function $A(\varphi)$ as

$$S = \frac{1}{16\pi G}(S_H + S_\varphi) + S_M[\psi_M, A^2(\varphi)g_{\mu\nu}] \tag{C1}$$

with S_H is the hilbert action, S_M is the matter piece of the action and is a functional of matter variables ψ_M and the Jordan-Fiertz metric $\tilde{g}^{\mu\nu} = A^2(\varphi)g_{\mu\nu}$ S_φ is the scalar piece of the action and is given as

$$S_\varphi = \int d^4x\sqrt{-g}(-2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi)$$

Again, we demand variation of the action with respect to the inverse metric to vanish

$$\begin{aligned}0 =& \delta S \\ =& \int d^4x\sqrt{-g}\Bigg(\frac{1}{16\pi G}\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} + \frac{1}{16\pi G}\frac{1}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta g^{\mu\nu}} \\ &+ \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\Bigg)\delta g^{\mu\nu}\end{aligned}$$

From the first and the last term we get the same result as in the ordinary general relativity(Eq. B2 and Eq. B1). Therefore we examine the second one. Using $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta(g^{\mu\nu})$

$$\delta S_\varphi = -2\int d^4x\Big[\delta(\sqrt{-g})g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \sqrt{-g}\partial_\mu\varphi\partial_\nu\varphi\delta(g^{\mu\nu})\Big]$$
$$= -2\int d^4x\Big[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta(g^{\mu\nu})g^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi + \partial_\mu\varphi\partial_\nu\varphi\delta(g^{\mu\nu})\sqrt{-g}\Big]$$
$$= \int d^4x\sqrt{-g}\Big[g_{\mu\nu}g^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi - 2\partial_\mu\varphi\partial_\nu\varphi\Big]\delta(g^{\mu\nu}) \tag{C2}$$

Therefore we end up with

$$\frac{1}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta g^{\mu\nu}} = g_{\mu\nu}g^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi - 2\partial_\mu\varphi\partial_\nu\varphi$$

Putting everything together we have the equations of motions as

$$G_{\mu\nu} = -g_{\mu\nu}g^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi + 2\partial_\mu\varphi\partial_\nu\varphi + 8\pi G T_{\mu\nu} \tag{C3}$$

or alternatively by taking the trace we have

$$R = 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 8\pi G T$$

which leads to

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{C4}$$

Now to write the equation of motion for φ field we look for variation of the action with respect to φ

$$0 = \delta S$$
$$= \int d^4x\sqrt{-g}\Big(\frac{1}{16\pi G}\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta\varphi} + \frac{1}{16\pi G}\frac{1}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta\varphi} + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta\varphi}\Big)\delta\varphi \tag{C5}$$

$\frac{\delta S_H}{\delta\varphi} = 0$ because there no φ dependence of S_H. Also, $\frac{\delta S_\varphi}{\delta\varphi}$ gives the euler-lagrange equations from which we get

$$\frac{1}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta\varphi} = 4\Box\varphi \tag{C6}$$

Now, we calculate the variation of S_M with respect to φ. Using $g^{\mu\nu}A(\varphi)^{-2} = \tilde{g}^{\mu\nu}$ and $\tilde{T}_{\mu\nu} = A(\phi)^{-2}T_{\mu\nu}$ and $\sqrt{\tilde{g}} = A^4\sqrt{g}$ we get

$$\frac{\delta S_M}{\delta\varphi} = \frac{\delta S_M}{\delta\tilde{g}^{\mu\nu}}\frac{\partial\tilde{g}^{\mu\nu}}{\partial\varphi}$$
$$= \Big(-\frac{\sqrt{-\tilde{g}}}{2}\tilde{T}_{\mu\nu}\Big)\Big(\frac{\partial(A^{-2})}{\partial\varphi}g^{\mu\nu}\Big)$$
$$= \Big(-\frac{A^4\sqrt{-g}}{2}A^{-2}T_{\mu\nu}\Big)\Big(\frac{-2}{A^3}\frac{\partial A}{\partial\varphi}g^{\mu\nu}\Big)$$
$$= \sqrt{-g}\alpha(\varphi)T \tag{C7}$$

Where we defined $\alpha(\varphi) = \partial\ln A/\partial\varphi$. Putting Eq. (C6) and (C7) together in Eq. (C5) we have

$$0 = \delta S = \int d^4x\sqrt{-g}\Big(\frac{1}{16\pi G}4\Box\varphi + \frac{1}{\sqrt{-g}}\sqrt{-g}\alpha(\varphi)T\Big)\delta\varphi \tag{C8}$$

Rearranging we have

$$\Box\varphi = -4\pi G\alpha(\varphi)T \tag{C9}$$

Equation C4 together with C9 are the equations of motions for the action (C1).

2. Massive Scalar Field

In massive case, as we discussed earlier we have a very similar Action as in C1. The only difference is the addition of a mass term in S_φ, hence

$$S_\varphi = \int d^4x\sqrt{-g}(-2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 2m^2\varphi^2)$$

Therefore only difference will come from the variation of S_φ. First looking at the variation with respect to metric we have using $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta(g^{\mu\nu})$

$$\delta S_\varphi = \int d^4x\sqrt{-g}\Big[g_{\mu\nu}g^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi - 2\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2 g_{\mu\nu}\Big]\delta(g^{\mu\nu}) \tag{C10}$$

Therefore, the equations of motion for metric becomes

$$R_{\mu\nu} = 8\pi\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big) + 2\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2 g_{\mu\nu} \tag{C11}$$

With very similar fashion, varaiation of S_φ with respect to scalar field φ using Euler-Lagrange equations turns out to be

$$\frac{1}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta\varphi} = 4\Box\varphi - 4m^2\varphi \tag{C12}$$

Hence the equations of motion for φ becomes

$$\Box\varphi = -4\pi\alpha(\varphi)T + m^2\varphi \tag{C13}$$

where all things are defined as before.