PHYS516 General Relativity and Astrophysics Project: Quantum Field Theory in Curved Spacetime and Unruh Effect

Ekrem S. Demirboğa

Department of Physics, Koç University,

Rumelifeneri Yolu, 34450 Sariyer, Istanbul, Turkey

(Dated: May 31, 2021)

This is the term project of the course PHYS516. In this paper, I will discuss the topic "Quantum Field Theory (QFT) in Curved Spacetime". I will mostly follow the discussion in Carroll(2019) Chapter 9 and Introductory Notes from Frodden and Valdés as well as lecture notes of Tong and Jacobson and Ford . I will start with a brief review of Quantum Mechanics with Heisenberg Picture (QM) and an introduction to QFT in in Flat Spacetime. Then, I will generalize the idea and discuss the QFT in Curved Spacetime. Finally, I will discuss a prediction of QFT which is the so called 'Unruh Effect'.

QUANTUM MECHANICS AND HEISENBERG PICTURE

In Classical Mechanics the state of the system is described as points in phase space which usually written as (q_i, p_i). On the other hand, in Quantum Mechanics the state of the system is described by the state vector ψ living in Hilbert Space. To move from a classical system to a quantum mechanical system, we usually promote the functions f on phase space to the operators $\hat{f}$ on Hilbert Space. The map between these to is given by the relation between the Poisson brackets and commutators.

$$\{,\} \rightarrow -\frac{i}{\hbar}[,] \tag{1}$$

This is called *canonical quantization.* Particularly we have

$$[\hat{q}_a, \hat{q}_b] = [\hat{p}_a, \hat{p}_b] = 0 \text{ and } [\hat{q}_a, \hat{p}_b] = i\hbar\delta_a^b$$

In classical mechanics dynamics of the system is governed by Hamiltonian and Hamilton' equations. In quantum mechanics, the dynamics of the system is again governed by Hamiltonian, but this time there are two ways to picture the evolution of the system. These are called *Schrödinger Picture* and *Heisenberg Picture.* Schrödinger picture represents the evolution of the system as unitary evolution of the state vector in Hilbert Space and the state vector obeys the Schrödinger Equation

$$H\psi = i\partial_t\psi \tag{2}$$

where we took $\hbar = 1$. Heisenberg picture however, the state is kept fixed and the observables evolve in time obeying the Heisenberg Equation of motion

$$\frac{d\hat{A}(t)}{dt} = i[H, \hat{A}(t)] \tag{3}$$

where A is some observable. In other words, we have the relations

$$\begin{aligned} |\psi\rangle_H =& U^\dagger |\psi(t)\rangle_S \\ \hat{A}(t)_H =& U \hat{A}_S U^\dagger \end{aligned}$$

where subscripts added for showing which picture we are in. It is important to note that these different pictures are completely equivalent.

One special example to study in this framework is the simple harmonic oscillator. One way to solve this is by introducing creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ as

$$\hat{a} = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} + i\hat{p}), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} - i\hat{p})$$

with the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{4}$$

And Hamiltonian of the system is

$$\hat{H} = -\frac{1}{2}\partial_x^2 + \frac{1}{2}\omega^2\hat{x}^2 = (\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega$$

we also define the number operator $n \equiv \hat{a}^\dagger\hat{a}$ which basically counts the number of exication from the ground state when acted upon a state. Then, we can write Heisenberg Equations of Motion as

$$\frac{d\hat{a}}{dt} = -i\omega\hat{a}, \qquad \frac{d\hat{a}^\dagger}{dt} = i\omega\hat{a}^\dagger$$

which yields the solutions

$$\hat{a}(t) = e^{-i\omega t}\hat{a}(0), \qquad \hat{a}(t)^\dagger = e^{i\omega t}\hat{a}(0)^\dagger$$

from which we can write position and momentum operators as

$$\begin{aligned} x(t) &= x(0)cos(\omega t) + \frac{p(0)}{m\omega}sin(\omega t) \\ p(t) &= p(0)cos(\omega t) - m\omega x(0)sin(\omega t) \end{aligned} \tag{5}$$

from which we can calulate the expectations

$$\langle x(t)\rangle = \langle x(0)\rangle cos(\omega t) + \frac{\langle p(0)\rangle}{m\omega}sin(\omega t)$$

If we were to do the calculations in Schrödinger Picture, we would express the states as combination of number eigenvectors: $|\psi\rangle = \sum_n c_n |n\rangle$. Therefore evolution would become simply $|\psi\rangle = \sum_n c_n e^{-iE_n t} |n\rangle$. And if we were to calculate the expectation value $\langle x\rangle$ as

$$\langle x\rangle = \sum_{m,n} \langle\psi_n|x|\psi_m\rangle = c_n c_m^* \langle m|x|n\rangle e^{-i(E_n-E_m)t}$$

since only nonzero terms in the sum are $m = n \pm 1$, hence, we get the same results as Heisenberg picture.

QFT IN FLAT SPACETIME

Quantum Field Theory (QFT) just like quantum harmonic oscillator, is just a quantum mechanical system in which we will be quantizing fields rather than a single oscillator. And there are reasons to think that fields are more fundamental than particles. First of all, the primary reason to introduce fields in physics is related to the fact that Laws of Nature are local. Other than that, Special Relativity and Quantum Mechanics together implies that particle number is not conserved. We will see this more clearly when we introduce the Unruh Effect but a more simply way to why this is true is the Heisenberg Uncertainty Principle. We know that a particle trapped in a box with length L has an uncertainty in its momentum as $\Delta p \geq \hbar/L$. In relativity we know that momentum and energy are on equal footing. Therefore this implies that there is an uncertainty in energy and when this uncertainty becomes comparable with the rest mass of the particle, particle-anti particle pairs popping out of vacuum will be important. This happens when a particle localized into a distance $\lambda = \hbar/mc$ which is known as *Compton Wavelength*. Just like de Broglie wavelength tells when the wavelike properties of particle becomes important, Compton wavelength tells when the concept single point-like particle becomes vague[1]. Finally we know that all fundamental particles of the same type in Standard Model are the same. Therefore, it is natural think of each particle of the same type have the same origin, i.e., field. Therefore in this section we will be quantizing fields, in particular, a scalar field in Minkowski Spacetime. Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \tag{6}$$

Then, Euler-Lagrange equations yields the Klein-Gordon Equation

$$\Box\phi - m^2\phi = 0 \tag{7}$$

[1] For more discussion check Tong QFT Lecture notes

We also want to study the Hamiltonian picture, hence, we define conjugate momentum field,

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \dot{\phi} \tag{8}$$

We also have Hamiltonian density, just like Lagrangian density we will be referring it as Hamiltonian and it is given Legendre transformation by

$$\mathcal{H}(\phi, \pi) = \pi\dot{\phi} - \mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{9}$$

where $(\boldsymbol{\nabla}\phi)^2 = \delta^{ij}\partial_i\partial_j$. Now we would like to write down the solutions to Klein Gordon Equation. A generic solution can be written as

$$\phi(x_\mu) = \phi_0 e^{k\mu x^\mu} \tag{10}$$

where $\omega^2 = \boldsymbol{k}^2 + m^2$. It can be seen clearly that there is a similarity between harmonic oscillator solution to Klein Gordon equation. In fact, Solutions to the Klein Gordon can be thought of as linear superposition of harmonic oscillator solution each with different frequency and this time frequency depends on $\boldsymbol{k}$.

Now in order to write the most general solution by constructing an orthonormal,complete set of modes, we define an inner product expressed as an inner product over a constant time hypersurface Σ_t,

$$(\phi_1, \phi_2) = -i\int_{\Sigma_t} (\phi_1 \partial_t \phi_2^* - \phi_2^* \partial_t \phi_1) d^{n-1}x \tag{11}$$

This integral is independent of Σ_t. To see this, consider a volume M contained in a surface $\Sigma = \Sigma_1 \cup \Sigma_2$ in Minkowski Spacetime. Then from Stoke's Theorem

$$\int_{\Sigma_1} d^{n-1}x n^\mu j_\mu + \int_{\Sigma_2} d^{n-1}x n^\mu j_\mu = \int_M d^n x \nabla^\mu j_\mu \tag{12}$$

For an arbitrary vector j_μ. Letting $j_\mu = i(\phi_1\partial_\mu\phi_2^* - \phi_2^*\partial_\mu\phi_1)$, we have $\nabla^\mu j_\mu = i(\phi_1^*\Box\phi_2 - \phi_2^*\Box\phi_1) = 0$. Now, Assume Σ_2 is in future of Σ_1 then, $n^\mu|_{\Sigma_1} = (1, 0, \dots)$ and $n^\mu|_{\Sigma_2} = (-1, 0, \dots)$ Therefore we have,

$$\begin{aligned}
\int_{\Sigma_1} d^{n-1}x j_0 - &\int_{\Sigma_2} d^{n-1}x j_0 = 0 \\
\int_{\Sigma_1} d^{n-1}x(\phi_1\partial_t\phi_2^* - \phi_2^*\partial_t\phi_1) - &\int_{\Sigma_2} d^{n-1}x(\phi_1\partial_t\phi_2^* - \phi_2^*\partial_t\phi_1) = 0 \\
\implies (\phi_1, \phi_2)_{\Sigma_1} = &(\phi_1, \phi_2)_{\Sigma_2}
\end{aligned} \tag{13}$$

Hence, the inner product is independent of the hypersurface. Now, we calculate the inner product for two solutions

$$
\begin{aligned}
(e^{ik_1^\mu x_\mu}, e^{ik_2^\nu x_\nu}) =& -i\int_\Sigma (e^{ik_1^\mu x_\mu}\partial_t e^{ik_2^\nu x_\nu} - e^{ik_2^\nu x_\nu}\partial_t e^{ik_1^\mu x_\mu}) \\
=&(\omega_1+\omega_2)e^{-i(\omega_1-\omega_2)t}\int_\Sigma e^{i(\boldsymbol{k_1}-\boldsymbol{k_2})\cdot\boldsymbol{x}} d^{n-1}x \\
=&(\omega_1+\omega_2)e^{-i(\omega_1-\omega_2)t}(2\pi)^{n-1}\delta^{n-1}(\boldsymbol{k_1}-\boldsymbol{k_2})
\end{aligned}
\tag{14}
$$

This shows that inner product vanishes when wave vectors $\boldsymbol{k}$ are different. From this we can construct an orthonormal set of mode like

$$
f_k(x^\mu) = \frac{e^{ik\mu x^\mu}}{\sqrt{(2\pi)^{n-1}2\omega}}
\tag{15}
$$

such that

$$
(f_{\boldsymbol{k}}, f_{\boldsymbol{k'}}) = \delta^{n-1}(\boldsymbol{k}-\boldsymbol{k'})
\tag{16}
$$

From the relation $\omega^2 = \boldsymbol{k}^2 + m^2$ for each given $\boldsymbol{k}$ we have $\pm\omega$. Therefore, at this point we choose the positive ω values only. To complete the set of modes given above we introduce the complex conjugate f_k^* which we will call negative frequency modes and f_k will be the positive frequency modes. In other words, positive modes satisfy

$$
\partial_t f_k = -i\omega f_k
$$

where negative modes satisfy

$$
\partial_t f_k^* = i\omega f_k^*
$$

provided $\omega > 0$. Then, we have

$$
(f_{\boldsymbol{k}}^*, f_{\boldsymbol{k'}}^*) = -\delta^{n-1}(\boldsymbol{k}-\boldsymbol{k'}) \qquad (f_k^*, f_{k'}) = 0
\tag{17}
$$

f's and f^*'s together form a complete set which we can write any solution in terms of them.

Now, we want to quantize this field. To do that we again introduce canonical quantization just like in (2). We promote the field and its conjugate momenta to operators in Hilbert Space with the following commutation relations

$$
\begin{aligned}
[\phi(t,x), \phi(t,x')] =& 0 \\
[\pi(t,x), \pi(t,x')] =& 0 \\
[\phi(t,x), \pi(t,x')] =& i\delta^{n-1}(x-x')
\end{aligned}
\tag{18}
$$

We can also expand the quantum operator field ϕ just like classical fields

$$\phi(x^\mu) = \int d^{n-1}x\Big(\hat{a}_k A_k e^{i(\boldsymbol{k}\cdot x-\omega t)} + \hat{b}_k B_k e^{i(\boldsymbol{k}\cdot x+\omega t)}\Big).$$

where we chose ω to be positive and divide the solution into two parts corresponding to the positive and negative roots of $\omega^2 = \boldsymbol{k}^2 + m^2$ which corresponds to propagating modes in the $\boldsymbol{k}$ and $-\boldsymbol{k}$ spatial directions. However, note that $\boldsymbol{k} \to -\boldsymbol{k}$ leaves ω invariant, so we can write the second term as $\hat{b}_{-k}B_{-k}e^{-i(\boldsymbol{k}\cdot x+\omega t)}$. If we impose the hermitian condition on ϕ we can write it as

$$\phi = \int d^{n-1}\left[\hat{a}_k f_k + \hat{a}_k^\dagger f_k^*\right] \tag{19}$$

Putting this expansion into the commutation relations give in (18) yields the following relations:

$$\begin{aligned}
[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}] =& 0\\
[\hat{a}_{\boldsymbol{k}}^\dagger, \hat{a}_{\boldsymbol{k}'}^\dagger] =& 0\\
[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger] =& \delta^{n-1}(\boldsymbol{k} - \boldsymbol{k}')
\end{aligned} \tag{20}$$

which is very similar to what we have for creation and annihilation operators in (4). This time, however, we have infinite amount of them. Just like in the harmonic oscillator case, we can write a state with n_i exication as each with momenta $\boldsymbol{k}_i$ as

$$|n_1, \ldots, n_i\rangle = \frac{1}{\sqrt{n_1! \ldots n_i!}}(a_{\boldsymbol{k}_1}^\dagger)^{n_1} \cdots (a_{\boldsymbol{k}_i}^\dagger)^{n_i}|0\rangle \tag{21}$$

Also we can define a number operator each wave vector as

$$\hat{n0}_{\boldsymbol{k}} = \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} \tag{22}$$

Then, we can form a basis for the Hilbert Space known as Fock Basis from the eigenstates of the number operator.

Now we want to express the Hamiltonian in terms of annihilation and creation operators just as in Harmonic Oscillator. Hamiltonian can be written as

$$H = \int d^{n-1}\mathcal{H} = \int d^{n-1}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right] \tag{23}$$

First, investagte the ϕ^2 term,

$$\begin{aligned}
\frac{1}{2}m^2\int d^{n-1}x\phi^2 =& \frac{1}{2}m^2\int d^{n-1}xd^{n-1}kd^{n-1}k'(\hat{a}_k f_k + \hat{a}_k^\dagger f_k^*)(\hat{a}_{k'} f_{k'} + \hat{a}_{k'}^\dagger f_{k'}^*)\\
=& \frac{1}{2}m^2\int d^{n-1}k\left(\frac{1}{2\omega}\right)\left[\hat{a}_k\hat{a}_{-k}e^{-2i\omega t} + \hat{a}_k^\dagger\hat{a}_k + \hat{a}_k\hat{a}_k^\dagger + \hat{a}_k^\dagger\hat{a}_{-k}^\dagger e^{2\omega t}\right]
\end{aligned} \tag{24}$$

where in the last line we Evaluated every terms in $\boldsymbol{k}'$ and x integral by plugging in f_k's from their definition. Similarly we can evaluate the other terms. kinetic energy term and gradient energy term will just give factors of ω and $\boldsymbol{k}$ coming from the derivatives,

$$\frac{1}{2}\int d^{n-1}x\dot{\phi}^2 = \frac{1}{2}\int d^{n-1}k\left(\frac{\omega}{2}\right)\left[-\hat{a}_k\hat{a}_{-k}e^{-2i\omega t} + \hat{a}_k^\dagger\hat{a}_k + \hat{a}_k\hat{a}_k^\dagger - \hat{a}_k^\dagger\hat{a}_{-k}^\dagger e^{2\omega t}\right] \tag{25}$$

and

$$\frac{1}{2}\int d^{n-1}x(\boldsymbol{\nabla}\phi)^2 = \frac{1}{2}\int d^{n-1}k\left(\frac{\boldsymbol{k}^2}{2}\omega\right)\left[-\hat{a}_k\hat{a}_{-k}e^{-2i\omega t} + \hat{a}_k^\dagger\hat{a}_k + \hat{a}_k\hat{a}_k^\dagger - \hat{a}_k^\dagger\hat{a}_{-k}^\dagger e^{2\omega t}\right] \tag{26}$$

Now, using the identity $\omega^2 = \boldsymbol{k}^2 + m^2$ we arrived our desired result

$$\begin{aligned} H =& \frac{1}{2}\int d^{n-1}k\left[\hat{a}_k^\dagger\hat{a}_k + \hat{a}_k\hat{a}_k^\dagger\right]\omega \\ =& \int d^{n-1}k\left[\hat{n}_k + \frac{1}{2}\delta^{n-1}(0)\right]\omega \end{aligned} \tag{27}$$

where we used the definition of number operator and the commutation relations (18). However, the delta function evaluated at 0 has infinite value as well as the integral over $\boldsymbol{k}$ has infinite range. But if we use discrete solutions we can get rid of the infinity arises from delta function. For example we can say that $[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}$. This is equivalent to restricting our Spacetime to a finite volume. But there is another infinity due to the range of our integral, $\boldsymbol{k}$. In order to get rid of this infinity, we must put a cutoff at some high value of $\boldsymbol{k}$.

However, just as in Harmonic oscillator case, we can redefine our Hamiltonian such that the ground state will give 0 energy. In other words, we can just subtract the infinity from Hamiltonian and quantize it that way. In this way, we can define a hamiltonian as

$$H = \int d^{n-1}k\hat{n}_k\omega \tag{28}$$

This technique is called *Renormalization.*

QFT IN CURVED SPACETIME

In this section we will follow the same discussion in previous section and will generalize to curved Spacetime. Let's start with the Lagrangian density

$$\mathcal{L} = \sqrt{g}\left(\frac{-1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla\nu\phi - \frac{1}{2}m^2\phi^2 - \xi R\phi^2\right) \tag{29}$$

which is almost the same as before, the only difference is the appearance of metric $g_{\mu\nu}$ and addition of coupling to curvature scalar. The coupling parameterized by ξ. $\xi = 0$ is called minimal coupling and $\xi = (n-2)/4(n-1)$ is called conformal coupling because it leaves the lagraningian invariant under conformal transformations.

This time conjugate momenta is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\nabla_0 \phi)} = \sqrt{-g}\nabla_0\phi \tag{30}$$

and canonical commutation relations are

$$\begin{aligned} [\phi(t,x), \phi(t,x')] =& 0 \\ [\pi(t,x), \pi(t,x')] =& 0 \\ [\phi(t,x), \pi(t,x')] =& \frac{i}{\sqrt{-g}}\delta^{n-1}(x-x') \end{aligned} \tag{31}$$

The equation of motion for the scalar field is

$$\Box\phi - m^2\phi - \xi R\phi = 0 \tag{32}$$

We again define an inner product over a spacelike surface Σ as

$$(\phi_1, \phi_2) = -i\int_\Sigma (\phi_1\nabla_\mu\phi_2^* - \phi_2^*\nabla_\mu\phi_1)n^\mu\sqrt{\gamma}d^{n-1}x \tag{33}$$

where γ^{ij} is the induced metric and n^μ is the unit normal vector on the Σ and the integral is independent of Σ[2]. At this point, just like in flat spacetime we would like to introduce a set of positive and negative frequency solutions and expand ϕ in terms of it. But there isn't necessarily a timelike killing vector, hence we may not be able to find solutions in which we can separate time and space dependent parts. Nevertheless, we can always find a set of solutions that are orthonormal which will not be unique. We know that the notion of number operator and Fock vacuum depends on the choice of set. In general, there does not exist a unique vacuum state in a curved spacetime. Therefore, the concept of particles becomes ambiguous, and the problem of the physical interpretation becomes much more difficult. Let's start by choosing such a set of orthonormal solutions,

$$(f_i f_j) = \delta_{ij} \qquad \text{and} \qquad (f_i^* f_j^*) = -\delta_{ij} \tag{34}$$

the index i can be continuous or discrete. For simplicity we assume discrete. Then, we can expand the field as

$$\phi = \sum_i \left(\hat{a}_i f_i + \hat{a}_i^\dagger f_i^*\right) \tag{35}$$

[2] the proof is similar to flat spacetime case

Hence, we have the following commutation relationss for $\hat{a}_i$ and $\hat{a}_i^\dagger$

$$
\begin{aligned}
[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}] =& 0 \\
[\hat{a}^\dagger_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}] =& 0 \\
[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}] =& \delta_{ij}.
\end{aligned}
\tag{36}
$$

Therefore, $hata_i$ and $\hat{a}_i^\dagger$ are annihilation and creation operators and there is a vacuum state defined by

$$\hat{a}_i \left|0_f\right\rangle = 0 \qquad \text{for all } i \tag{37}$$

Now from this vacuum state we can consturct a Fock Basis as before. And we can write a state with n_i exicatition as

$$\left|n_i\right\rangle = \frac{1}{\sqrt{n_i!}} (\hat{a}_i^\dagger)^{n_i} \left|0_f\right\rangle \tag{38}$$

And we define a number operator for each mode as

$$\hat{n}_{fi} = \hat{a}_i^\dagger \hat{a}_i \tag{39}$$

where the subscript f's are referring to the fact that we used the set of modes f_i among many other option. So far, the set f_i seems to be working quite well. But the problem is any other set of modes g_i would also work out. For example we can expand our field in this basis

$$\phi = \sum_i \left(\hat{b}_i g_i + \hat{b}_i^\dagger g_i^*\right) \tag{40}$$

The coefficients again satisfies the same commutation relations (36). And there will be a vacuum state $\hat{b}_i \left|0_g\right\rangle = 0$. Hence, we can construct another Fock Basis and define number operator $\hat{n}_{gi} = \hat{g}_i^\dagger \hat{g}_i$.

Now, to see what's going on we expand the modes in terms of each other.

$$
\begin{aligned}
g_i &= \sum_j \alpha_{ij} f_j + \beta_{ij} f_j^* \\
f_i &= \sum_j \gamma_{ij} f_j + \lambda_{ij} f_j^*
\end{aligned}
\tag{41}
$$

Recall that modes are orthonormal, so $\alpha_{ij} = (g_i, f_j)$, $\beta_{ij} = -(g_i, f_j^*)$. Therefore, $\gamma_{ij} = \alpha_{ij}^*$ and $\lambda_{ij} = -\beta_{ij}$. Hence,

$$
\begin{aligned}
g_i &= \sum_j \alpha_{ij} f_j + \beta_{ij} f_j^* \\
f_i &= \sum_j \alpha_{ij}^* f_j - \beta_{ij} f_j^*
\end{aligned}
\tag{42}
$$

This transformation is known as *Bogolubov Transformation* and the matrices α_{ij} and β are Bogolubov coefficients. Now, we can expand the field ϕ both in f and g modes. And if we insert the Bogolubov Transformation into one of them we can arrive at the result giving the relation between the operators of each mode as,

$$\begin{aligned}\hat{a}_i &= \sum_j (\alpha_{ji}\hat{b}_j + \beta^*_{ji}\hat{b}^\dagger_j) \\ \hat{b}_i &= \sum_j (\alpha^*_{ij}\hat{a}_j - \beta^*_{ij}\hat{a}^\dagger_j)\end{aligned} \tag{43}$$

From this relations it is easy to see that annihilation operators for one observer are not the annihilation operators of the other observer. Rather, they mix both creation and annihilation operators. Therefore, we can expect for one observer's vacuum to have particles for another observer. Let's verify this explicitly. Assume a state is in f vacuum $|0_f\rangle$ in which no particles observed by an observer in who uses the f modes, i.e., $\langle \hat{n}_{fi}\rangle = \langle 0_f|\hat{n}_{fi}|0_f\rangle = 0$. Now, let's look at the same state from an observer who uses the g mode expansion

$$\begin{aligned}\langle 0_f|\hat{n}_{gi}|0_f\rangle &= \left\langle 0_f \middle| \hat{b}^\dagger_i \hat{b}_i \middle| 0_f \right\rangle \\ &= \left\langle 0_f \middle| \sum_{j,k} \left(\alpha_{ij}\hat{a}^\dagger_j - \beta_{ij}\hat{a}_j\right)\left(\alpha^*_{ik}\hat{a}_k - \beta^*_{ik}\hat{a}^\dagger_k\right) \middle| 0_f \right\rangle \\ &= \left\langle 0_f \middle| \sum_{j,k} \beta_{ij}\beta^*_{ik}\hat{a}_j\hat{a}^\dagger_k \middle| 0_f \right\rangle \\ &= \sum_{j,k} \beta_{ij}\beta^*_{ik} \left\langle 0_f \middle| \hat{a}^\dagger_k\hat{a}_j + \delta_{jk} \middle| 0_f \right\rangle \\ &= \sum_j \beta_{ij}\beta^*_{ij}\end{aligned} \tag{44}$$

Therefore we have

$$\langle \hat{n_{gi}}\rangle = \sum_j |\beta_{ij}|^2 \tag{45}$$

In summary, vacuum in one frame may have particles in other frames. As can be seen from (42) β coefficients measures the mixing of positive and negative modes of different observers. Hence, if it is nonzero they will not agree on vacuum state. Although, this is not true for observers related by a Lorentz transformation . In other words, vacuum is Lorentz invariant but the problem arises when we have more general transformations.

At this point it is natural to ask whether our particle definition is correct. As Unruh showed [1], particle detectors use their proper times to define positive and negative frequency modes and therefore the particles,

$$\frac{D}{d\tau}f_i = -i\omega f_i \tag{46}$$

we use these modes to determine how many particle will a detector detect. However, this may not be possible all over spacetime but if we consider a static spacetime, there will be a timelike killing vector K^μ since metric components will be independent of time coordinate in some specific coordinate. In this case d'Alembertian becomes

$$\Box f = \left[g^{00}\partial_0^2 + \frac{1}{2}g^{00}g^{ij}(\partial_i g_{00})\partial_j + g^{ij}\partial_i\partial_j - g^{ij}\Gamma^k_{ij}\partial_k\right]f \tag{47}$$

Then, the equation of motion 32 can be written as

$$\partial_0^2 f = -(g^{00})^{-1}\left[\frac{1}{2}g^{00}g^{ij}(\partial_i g_{00})\partial_j + g^{ij}\partial_i\partial_j - g^{ij}\Gamma^k_{ij}\partial_k - (m^2+\xi R)\right]f \tag{48}$$

where LHS is only time derivative and RHS is spatial derivatives. Hence, we can find solutions of the form

$$f_\omega(t,x) = e^{-i\omega t}\bar{f}_\omega(x) \tag{49}$$

Now we can write, f is a positive frequency mode if

$$\partial_t f_\omega = -i\omega f_\omega, \qquad \omega > 0 \tag{50}$$

In coordinate invariant way, this is equivalent to

$$\mathcal{L}_K f_\omega = K^\mu\partial_\mu f_\omega = -i\omega f_\omega \tag{51}$$

and if f is negative frequency mode if

$$\mathcal{L}_K f^*_\omega = K^\mu\partial_\mu f^*_\omega = i\omega f^*_\omega \tag{52}$$

UNRUH EFFECT

As discussed in the previous section, vacuum state for one observer is not necessarily a vacuum state for another observer. Different observer uses different modes ,hence, different notion of vacuum and particles. Consider a Schrödinger particle in a box initially at ground state. As the box starts to accelerate, there is nonzero probability that the particle will be in an excited state. This example is a specific example of the *Unruh Effect*. In this section we will investigate this phenomenon. For simplicity we will compare two observers in flat space while one is accelerated.

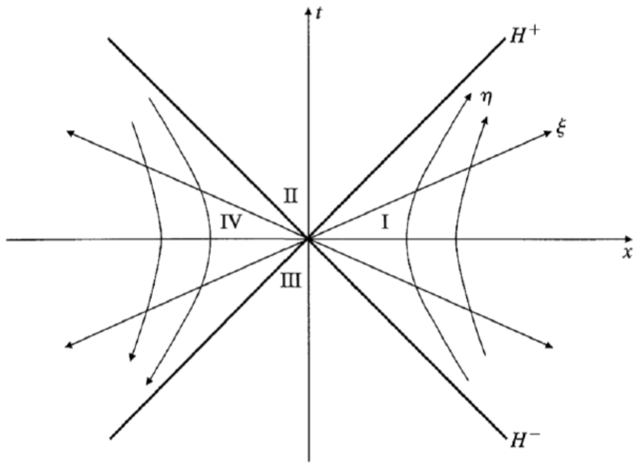

FIG. 1. Regions of Minkowski Spacetime (from Carrol)

We will also work in 2 dimension and take $m = 0$ and $\xi = 0$ for the most simple case. Therefore the the equation of motion 32 reduces to

$$\Box\phi = 0 \tag{53}$$

We are in flat spacetime so the metric is simply $\eta^{\mu\nu}$. Consider an observer moving with constant acceleration α, i.e., $\alpha = \sqrt{a^\mu a_\mu}$ where a $= d^2x^\mu/d\tau^2$. Now, it is easy to verify that the trajectory of such an observer is

$$\begin{aligned} t(\tau) =& \frac{1}{\alpha}\sinh(\alpha\tau) \\ x(\tau) =& \frac{1}{\alpha}\cosh(\alpha\tau) \end{aligned} \tag{54}$$

From this it is easy to calculate proper accelaaraiton

$$a^\mu = \frac{d^2x^\mu}{d\tau^2} = (\alpha\sinh(\alpha\tau), \alpha\cosh(\alpha\tau)) \tag{55}$$

The trajoctary describes an hyperboloid asympoting to null paths as in figure (1). Now we choose another coordinates for accelerated observer that cover the region I as

$$t = \frac{1}{a}f(\xi)\sinh(a\eta) \qquad x = \frac{1}{a}f(\xi)\cosh(a\eta) \tag{56}$$

This time we have

$$x^2 - t^2 = \frac{f(\xi)^2}{a^2} \tag{57}$$

In other words, observer cooresponds to coordinate ξ_0 has acceleration $a/f(\xi_0)$. Also, Observer with higer acceleration is closer to origin, hence, we require f to be inreasing function and choos $f(\xi) = e^{a\xi}$. In summary, metric has the form

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) \tag{58}$$

with $t \in \mathbb{R}$ and $x \in (t, \inf)$. Region I is called Rindler Space, and an observer moving along a constant acceleration path is called Rindler Observer.

Since metric components are independent of η we know that ∂_η is a killing vector. We can write it using chain rule as

$$\partial_\eta = \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x \tag{59}$$

And we can verify that it is timelike vector in Region I

$$(\partial_\eta)^\mu(\partial_\eta)_\mu = a^2(-x^2 + t^2) < 0 \tag{60}$$

Now, since we have a timelike killing vector we can use it to define our positive frequency modes by just saying if $\partial_\eta f = -i\omega t$ then f is positive. However, it is not easy to extend this into other regions of spacetime.

The equation of motion takes the form in Rindler coordinates

$$\Box\phi = e^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi = 0 \tag{61}$$

a plane wave $g_k = (4\pi\omega)^{-1/2}e^{-i\omega\eta + ik\xi}$ solves this equation in Region I with $\omega = |k|$. Apparently since $\partial_\eta g_k = -i\omega g_k$, it is a positive frequency mode. We can easily check that,

$$(g_k, g_{k'}) = i\int(g_k^*\partial_\eta g_{k'} - g_{k'}\partial_\eta g_k^*) = \delta(k - k') \tag{62}$$

$$(g_k^*, g_{k'}^*) = i\int(g_k\partial_\eta g_{k'}^* - g_{k'}^*\partial_\eta g_k) = -\delta(k - k') \tag{63}$$

$$\tag{64}$$

They are indeed orthonormal set.But when it comes to region IV things changes because, this coordinates are not valid there. But this is OK, we can just define a new coordinates by changing the sign of (56). But we don't need to introduce new coordinates since we only define them in their own region. In other words, when it comes to region IV we have a timelike killing vector

$\partial_{-\eta} = -\partial_\eta$ rather than ∂_η. Therefore, we introduce two sets of modes as

$$g_k^{(1)} = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi}, & \text{I} \\ 0, & \text{IV} \end{cases}$$
$$g_k^{(2)} = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{+i\omega\eta + ik\xi}, & \text{I} \\ 0, & \text{IV} \end{cases} \tag{65}$$

each mode is positive frequency with respect to their timelike killing vector. Now we have two different set of modes, Rindler and Minkowski. Therefore, just as in the previous section we need to calculate the Bogolubov Coefficients but there is an easier way. We will find a set of modes such that it will share the vacuum state with Minkowski vacuum. To do this we need to extend Rindler modes to whole spacetime in terms of original Rindler modes. First we write Rindler modes in terms of Minkowski coordinates

$$\begin{aligned} \sqrt{4\pi\omega} g_k^{(1)} &= e^{-i\omega(\eta-\xi)} = a^{i\omega/a}(-t+x)^{i\omega/a} \\ \sqrt{4\pi\omega} g_k^{(2)} &= e^{+i\omega(\eta+\xi)} = a^{-i\omega/a}(-t-x)^{-i\omega/a} \end{aligned} \tag{66}$$

Notice that they behave differently. But the combination

$$\sqrt{4\pi\omega}\big(g_k^1 + e^{-\pi\omega/a g_{-k}^{(2)*}}\big) = a^{i\omega/a}(-t+x)^{i\omega/a} \tag{67}$$

is well defined in both region I and IV which is the extension for$g_k^{(1)}$. By doing the same calculation for $g_k^{(2)}$, we define a new set of normalized modes as

$$h_k^{(1)} = \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\big(e^{\pi\omega/2a} g_k^{(1)} + e^{-\pi\omega/2a} g_{-k}^{(2*)}\big) \tag{68}$$
$$h_k^{(2)} = \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\big(e^{\pi\omega/2a} g_k^{(2)} + e^{-\pi\omega/2a} g_{-k}^{(1*)}\big) \tag{69}$$

Therefore, we can expand the field as

$$\phi = \int dk \Big(\hat{c}^{(1)} h_k^{(1)} + \hat{c}^{(1)\dagger} h_k^{(1)*} + \hat{c}^{(2)} h_k^{(2)} + \hat{c}^{(2)\dagger} h_k^{(2)*}\Big) \tag{70}$$

We have written the modes $\hat{h}$ in terms of modes g. Therefore, from our discussion of Bogolubov transformation, we know that we can also write the operators $\hat{c}$ in terms of operators $\hat{b}$ as

$$\begin{aligned} \hat{b}_k^{(1)} &= \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\big(e^{\pi\omega/2a}\hat{c}_k^{(1)} + e^{-\pi\omega/2a}\hat{c}_{-k}^{(2)\dagger}\big) \\ \hat{b}_k^{(2)} &= \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\big(e^{\pi\omega/2a}\hat{c}_k^{(2)} + e^{-\pi\omega/2a}\hat{c}_{-k}^{(1)\dagger}\big) \end{aligned} \tag{71}$$

Therefore we can express the number operator in Rindler space in terms of the new operators. In this new defined modes $\hat{c}_k$'s are asociated with positive frequency Minkowski modes, i.e., annihilation operators, while $\hat{c}_k^\dagger$'s associated with negative frequency Minkowski modes, i.e., creation operators, Thus,

$$\hat{c}_k^1 |0_M\rangle = \hat{c}_k^2 |0_M\rangle = 0 \tag{72}$$

Now we can calculate what the expected value of the number operator for a Rindler observer in Region I as

$$\begin{aligned}\left\langle 0_M \middle| \hat{n}_R^{(1)} \middle| 0_M \right\rangle &= \left\langle 0_M \middle| \hat{b}_k^{(1)\dagger} \hat{b}_k^{(1)} \middle| 0_M \right\rangle \\ &= \frac{1}{2\sinh(\pi\omega/a)} \left\langle 0_M \middle| e^{-\pi\omega/a} \hat{c}_{-k}^{(1)} \hat{c}_{-k}^{(1)\dagger} \middle| 0_M \right\rangle \\ &= \frac{e^{-\pi\omega/a}}{2\sinh(\pi\omega/a)} \delta(0) \\ &= \frac{1}{e^{2\pi\omega/a} - 1} \delta(0)\end{aligned} \tag{73}$$

The delta function comes from the fact our basis modes are not square-integrable. If we were to choose a finite spacetime the factor would be constant,

$$\langle n_R^{(1)} \rangle \sim \frac{1}{e^{2\pi\omega/a} - 1} \tag{74}$$

In Summary, We have shown that when an inertial observer sees a vacuum state, a non-inertial observer will see particles. In addition, if the non-inertial observer has a constant acceleration, i.e., if she is Rindler observer, the particles have a very unique distribution as in (74). This is Planckian thermal distribution with characteristic temperature called *Unruh Temperature*,

$$T = \frac{a}{2\pi} \tag{75}$$

which is linear in proper acceleration. Putting all the constants back we get the idea of the weakness of Unruh Effect

$$T = a \frac{\hbar}{2\pi c k_B} \approx 4 \times 10^{-21} a[K] \tag{76}$$

I would like to thank Andrew Coates, the instructor of the course for inspiring me on learning this subject.

[1] W. G. Unruh, Notes on black-hole evaporation, Phys. Rev. D **14**, 870 (1976).