Stars from Newton to Einstein, and Beyond

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In this project we calculated the various type of stars' structure in Newtonian gravity, general relativity and alternative theories of gravity which try to surpass the general relativity. We mainly studied White Dwarves (WDs) and then moved the Neutron Stars (NSs). We used Python for computational work as well as some Mathematica.

NEWTON

We start considering the hydro-static equilibrium of stars in Newtonian gravity. For a star in hydro static equilibrium, we have the following system of ODEs

$$\frac{dm(r)}{dr} = \pi r^2 \rho(r) \tag{1}$$
$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

where $m(r)$ is the mass within radius r, $\rho(r)$ is the density and $P(r)$ is the pressure.

To relate the pressure P and density ρ we use the equation of state for the stellar matter (EOS)

$$PV = NkT \implies P = \frac{k}{\mu m_H} T\rho \tag{2}$$

where T is temperature m_H is the mass of hydrogen atom and μ is the average molecular weight. For the following discussion we assume a poly-tropic EOS

$$P = K\rho^\gamma = K\rho^{1+\frac{1}{n}} \tag{3}$$

where n is the poly-tropic index. For stars with EOS in eq3, we can use the famous *Lane-Emden Equation.*

Lane-Emden Equation

Rearranging Eqs1 and differentiating gives

$$\frac{d}{dr}\left(\frac{1}{\rho}\frac{dP}{dr}\right) = \frac{2Gm(r)}{r^3} - \frac{G}{r^2}\frac{dm(r)}{dr}$$
$$= \frac{-2}{\rho r}\frac{dP}{dr} - 4\pi G\rho$$

Multiplying both sides with r^2 and rearranging yields

$$r^2\frac{d}{dr}\left(\frac{1}{\rho}\frac{dP}{dr}\right) + \frac{2r}{\rho}\frac{dP}{dr} = \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4Gr^2\rho$$

dividing both sides by r^2 we get the dimensional version of Lane-Emden Equation. Therefore, by substituting $\rho = \rho_c\theta^n$ or similarly $P = \rho_c^{1+\frac{1}{n}}\theta^{n+1}$ where ρ_c is the density of the center of the star, and scale the equation with

$$R = \alpha\xi \tag{4}$$

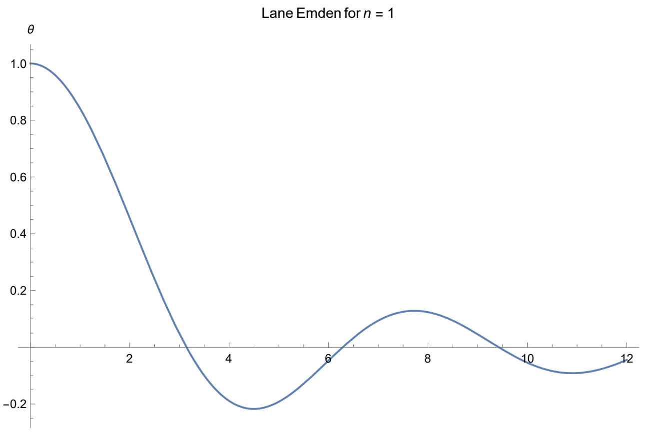

FIG. 1. Solution to the Lane-Emden Equation for n=1

where $\alpha = \frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}$ we get the Lane-Emden Equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{5}$$

We solved the equation around $\xi = 0$ using Mathematica[1] and found that there is two solution

$$\theta_1(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 + \ldots$$
$$\theta_2(\xi) = \frac{1}{\xi} - \frac{\xi}{2} + \frac{\xi^3}{24}\ldots$$

However, since the second solution diverges, it is not a physical solution. Therefore we continue with the first equation and we conclude that initial conditions are $\theta(0) = 1$ and $\theta'(0) = 0$. Again using Mathematica[2] we solved the IVP and found that for n = 1

$$\theta(\xi) = \frac{sin\xi}{\xi}$$

From the substitution that we made, we know that $\theta = 0 \iff \rho = 0$. We expect pressure to be zero hence,

[1] NewtonPartA.nb
[2] NewtonPartA.nb

density to be zero at the surface i.e. $P(R) = \rho(R) = 0$ where R is the radius of the star. Therefore we can write $\theta(\xi_n) = 0$ where $\alpha\xi_n = R$. Then we can write the the total mass M as

$$M = \int_0^R 4\pi r^2 \rho(r)dr = \int_0^{\xi_n} 4\pi\alpha^3\rho_c\xi^2\theta^n(\xi)d\xi \quad (6)$$

Using Eq.5 we can substitute θ^n and easily evaluate the integral

$$\begin{aligned} M =& -4\pi\alpha^3\rho_c \int_0^{\xi_n} \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi \\ =& 4\pi\alpha^3\rho_c(-\xi_n^2\theta'(\xi_n)) \\ =& 4\pi R^3\rho_c\left(\frac{\theta'(\xi_n)}{\xi_n}\right) \end{aligned} \quad (7)$$

So, we derived the total mass M(Eq.7) and we have an expression for R (Eq.4). Using these expressions and eliminating the ρ_c in both expression we relate the mass and the radius of stars

$$M = BR^{\frac{3-n}{1-n}} \quad (8)$$

where $B = 4\pi\left(\frac{K}{G}\frac{n+1}{4\pi}\right)^{n/n+1}\xi_n^{1+n/n-1}(-\theta'(\xi_n))$.

Since WDs are extremely dense objects their pressure is dominated by a quantum mechanical effect named electron degeneracy. Therefore EOS for cold WDs are give by

$$P = C[x(2x^2-3)(x^2+1)^{1/2} + 3sinh^{-1}x] \quad (9)$$

where $x = \left(\frac{\rho}{D}\right)^{\frac{1}{q}}$. We cannot use *Lane-Emden Equation* for these kind of stars. However, we can follow the same procedure and find an equation, modified version of *Lane-Emden Equation* called *Chandrasekhar White Dwarf Equation*.

Chandrasekhar White Dwarf Equation

We have the Hyrdrostatic Equlibrium Equation for a star(Eq.1) and the relation between the density and the pressure(Eq.9), therefore, we substitute pressure in the Hyrdrostatic Equlibrium Equations

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{d\sqrt{x^2+1}}{dr}\right) = -\frac{\pi GD^2}{2C}x^3$$

we define $y^2 = x^2 + 1$ and denote $\rho_c = Dx_c^3 = D(y_c^2 - 1)^{3/2}$. Also we define

$$r = \beta\eta \quad (10)$$

where $\beta = (\frac{2C}{\pi GD^2})^{1/2}\frac{\eta}{y_c}$ and finally we define $y = y_c\phi$. Then the equation reduces to

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right) + (\phi^2 - \frac{1}{y_c^2})^{3/2} = 0 \quad (11)$$

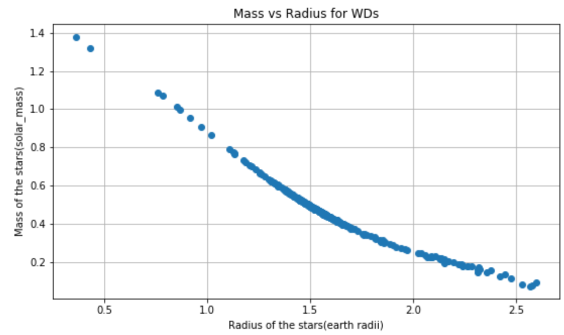

FIG. 2. M-R for given data

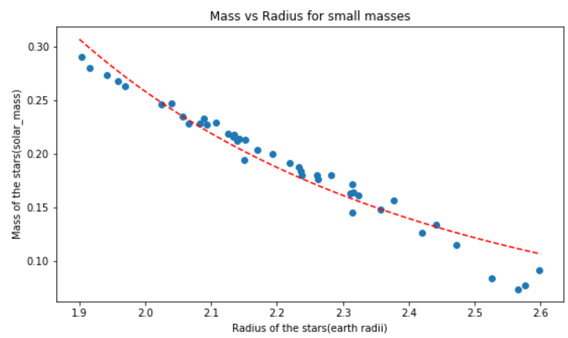

FIG. 3. fit to M-R data for low mass stars

which is called *Chandrasekhar White Dwarf Equation*. The initial conditions are similar to Lane-Emden. We have $\phi(0) = 1$ and $\phi'(0) = 0$ but at the surface $R = \beta\eta_n$ we have $\phi'(\eta_n) = \frac{1}{y_c^2}$.

M-R curve

We plotted the M-R points in figure 2 using the given data[3]. Then, for the law mass stars i.e for $x \ll 1$, using Mathematica[4] we showed that Eq.9 becomes

$$P = K_*\rho^{1+\frac{1}{n_*}} \quad (12)$$

where $K_* = \frac{8C}{5D^{5/q}}$ and $n_* = q/(5-q)$. Therefore, we can now use Lane-Emden and Eq.8. We made a fit to the data(figure3) and obtain the following values(in SI units) using Eq8.

$$\begin{aligned} K_* &= 3144530.473379261 \\ n_* &= 1,458045791 \\ q &\approxeq 2.96 \end{aligned}$$

[3] whitedwarfdata.csv
[4] NewtonPartB.nb

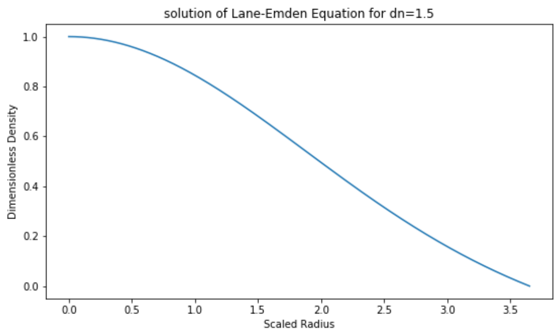

FIG. 4. Solution to the Lane-Emden for n =1.5

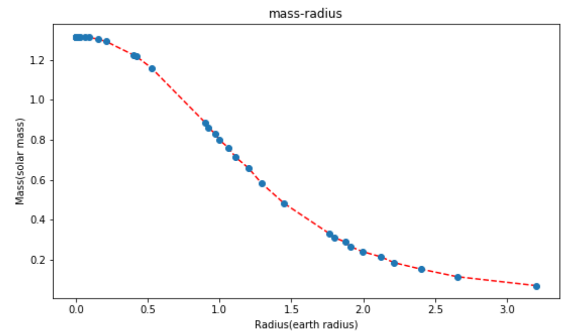

FIG. 6. M-R curve for WDs

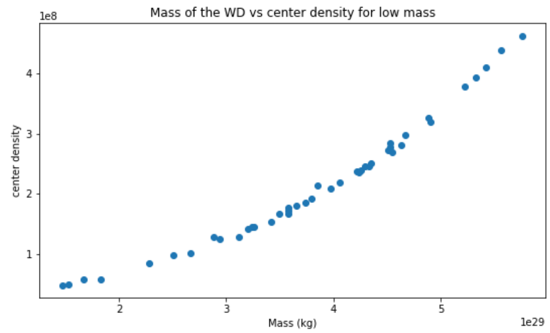

FIG. 5. Mass-ρ_c curve for low mass WDs

But we know from theory that q is an integer. Hence in order to make q an integer (i.e $q = 3$) we take $n = 1.5$. Since we know the index n we can now solve the Lane Emden equation. For $n = 1.5$ solution to the Lane-Emden(figure4) gives

$$\xi_n = 3.653680580580581$$
$$\theta'(\xi_n) = -0.20308599225966426$$

Since we know the ξ_n and $\theta'(\xi_n)$ we calculated the density of the center of the star ρ_c for each pair of $R - M$ using Eq 7(Figure5).

We found the parameter n and the relation between C and D from the low mass fit for Eq9. So, we have only one unknown parameter D. To find the correct value of D we did the following: We first start with an initial guess of $D = 1.7 \times 10^9$ looking at the figure(5) and the fact that $x = (\rho/D)^{1/3} \ll 1$ for low mass stars and rise up to unity for others. For this initial guess we chose 20 random rho_c values that will cover the whole R range in our data. Using the K value found we calculated the value of C. Then, with all the parameters we solved the *Chandrasekhar's WD Equation(Eq.11)*. From the solutions we calculated the corresponding R and M values, interpolated them using *spline* from scipy and calculated the error from the original data. We repeated the same procedure for different values of D to find the optimum value that minimizes the error. Eventually, we found the D and corresponding C with an error $2.1075048042135396 \times 10^{-9}$ as

$$C = 5.380892404212922 \times 10^{21} \quad (13)$$
$$D = 1830000000.0$$
$$K = 3144530.473379261$$

where the theoretical values of C and D is given as

$$C = \frac{m_e^4 c^5}{24\pi^2 \hbar^3} = 6.002332114024319 \times 10^{21} \quad (14)$$
$$D = \frac{m_u m_e^3 c^3 \mu_e}{3\pi^2 \hbar^3} = 1947865435.2624369$$
$$K = 3161125.6038212245$$

Chandrasekhar Mass Limit

Since we have all the parameters, we plotted the whole M-R curve(figure6) As can be seen in plot(figure6) there is a maximum mass allowed for WDs called *Chandrasekhar Mass Limit*. We approximately calculated the value.[5]. We started with some random ρ_c values. Then we calculated the corresponding mass for each ρ_c. Then we saw that as we increase the ρ_c we got a NS with a higher mass. So each time we increase the ρ_c we calculated the difference between the last two masses and we saw that the difference is converging to zero as expected and highest mass value converges to some number that is found as

$$M_{Ch} = 1.3178477203482308 M_\odot \quad (15)$$

We also plotted the convergence of this calculation in figure(7) We can also theoretically calculate this limit. Using Mathematica[6] we showed that for $x \gg 1$ the Eq9 becomes

$$P = 2Cx^4 + \dots \quad (16)$$

[5] Chandrasekhar.py
[6] mathematica folder

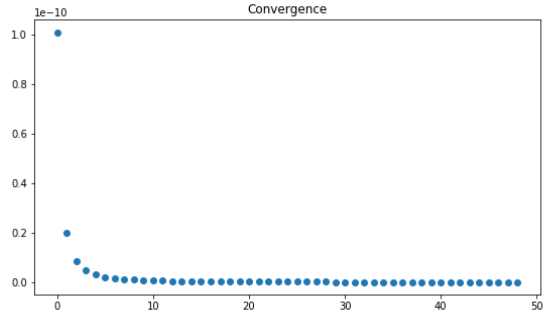

FIG. 7. Convergence of the Highest Mass Limit

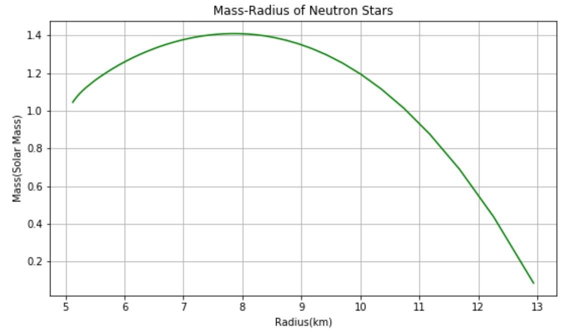

FIG. 8. M-R curve for NSs

or since we know that $q = 3$

$$P = K\rho^{\frac{4}{3}} \implies n = 3 \tag{17}$$

where $K = \frac{2C}{D^{4/3}}$. Then Using Equation 8 we can find the Chandrasekhar Mass Limit for WDs as

$$M_{Ch} = B = 4\pi\left(\frac{K}{G\pi}\right)^{3/2}\xi_3^2(-\theta'(\xi_3)) \tag{18}$$

$$\tag{19}$$

If we put the constants from Eq.14 as well as the solution of the Lane-Emden equation for n=3 which are found as $\xi_3 = 6.926992292292292$ and $\theta'(\xi_3) = -0.04210902023243969$ we found the Chandrasekhar Mass as

$$M_{Ch} = 1.45832101520284M_{\odot} \tag{20}$$

Which is relatively close to what we found.

EINSTEIN

We know that there is a mass limit for WDs and we calculated the value of it. Existence of such a limit brings the question: "what would happen if we try to add even more mass to WD?". The solutions we obtain so far was for WD at equilibrium. If a WD goes out of equilibrium it starts to collapse and instability continuous until some other mechanism takes place. Sometimes these collapses can end up in big explosions and destruction of WDs called *Type Ia supernova* explosions. And sometimes WD squeezes so much that electrons and protons merges and creates neutrons. If the electron degeneracy pressure can obtain stability, a star made of mostly neutrons emerges. These stars called as Neutron Stars (NS). NS usually have a radius around 10-20 km and a few solar mass. Therefore gravity becomes so immense that we can no longer use the Newton's Equations, instead we use Einstein's General Relativity.

We can expect NS to behave like WD but with a non-interacting electron model, however, neutrons in NS interacts with each other. In fact, the interaction are so complicated that we don't know the exact EOS. Therefore, for simplicity we will use

$$P = K_{NS}\rho^2 \tag{21}$$

which gives qualitative behavior of NSs. And for Simplicity we take $K = 50$

From now on, we will use the geometric units i.e. $M_{\odot}$ as the mass unit, $\frac{GM_{\odot}}{c^2} \approx 1477m$ as length unit and $\frac{GM_{\odot}}{c^3} \approx 4.927 \times 10^{-6}s$ as time unit so that $c = G = 1$.

Tolman-Oppenheimer-Volkoff (TOV) Equations

Hydrostatic equilibrium equations are modified due to general relativistic effects are called Tolman-Oppenheimer-Volkoff (TOV) Equations and given as:

$$m' = 4\pi r^2\rho \tag{22}$$
$$\nu' = 2\frac{m + 4\pi r^3 p}{r(r-2m)}$$
$$p' = -\frac{1}{2}(\rho + p)\nu'$$

where ' indicates the derivative w.r.t r and $e^{\nu(r)/2}$ is the time dilation factor of relativity. We solve this system of ODEs just as in Newtonian case. We will stop when $p = 0 \iff \rho = 0$ with initial conditions $m(0) = 0$ and $p(0) = p_c$ or $\rho(0) = \rho_c$. And by definition, $e^{v(0)/2}$ is the time dilation factor due to gravity at the center of the star compared to an observer at $r \to \infty$. So we need to start with $\nu(0)$ that gives $\mu(\infty) = 0$. We cannot know this in advance but it does not matter. Because adding a constant to μ does not effect the solution of $m(r)$ and $p(r)$. Therefore we will simply use $\mu(0) = 0$.

We obtained the $M - R$ curve(figure 8) by solving Eqs22 for different p_c's.

But, the $m(r)$ functions that we calculated is actually not the rest mass of the stars. Since mass is the energy

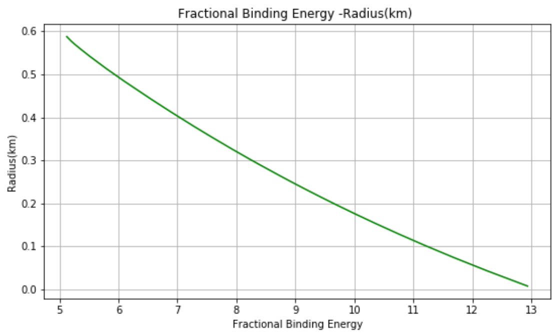

FIG. 9. Fractional Binding Energy - Radius curve of Neutron Stars

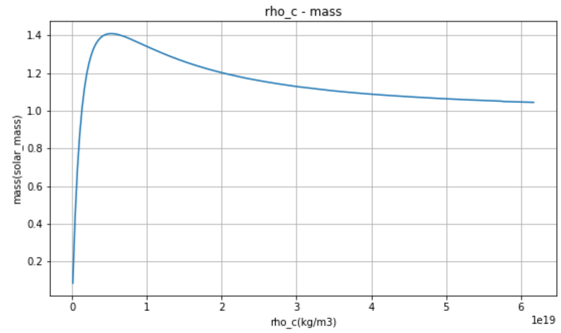

FIG. 10. $\rho_c vs M$ Curve

in relativity and gravitational potential energy is negative; what we calculated contains the rest mass as well as negative contribution due gravity. Therefore, in order to find the rest mass which is called *baryonic mass* we need to solve the following ODE

$$m'_P = 4\pi\Big(1 - \frac{2m}{r}\big)^{-1/2} r^2 \rho \tag{23}$$

And we define the fractional binding energy as

$$\Delta \equiv \frac{M_P - M}{M} \tag{24}$$

We plotted Gravitational potential energy Δ vs Radius R in figure 9

Stability of Neutron Stars

Stability condition for a NS is give as

$$\frac{dM}{d\rho_c} > 0 \rightarrow stable \tag{25}$$
$$\frac{dM}{d\rho_c} < 0 \rightarrow unstable$$

The reason behind these conditions are as follows. If we squeeze the star a little bit, a stable star resist this process since it is stable. Therefore we do positive work which means mass increases since mass is energy in relativity. Density also increases but overall $\frac{dM}{d\rho_c} > 0$. If star is unstable it continuous squeezing since it already wants to go to a lower energy level. It continuous doing this till it finds a stable equilibrium or forms a black hole.

Therefore we plotted the $\rho_c vs M$ curve(figure 10) to and find the stability region for the NSs. Then by looking for the conditions in Eq.25 we conclude where the NSs are stable and where they are unstable. We plotted to M-R curve and indicate which NSs are stable in figure 11. Also we conclude that there is a maximum mass M_{max} allowed by this EOS. From figure 11 or figure 8 we found

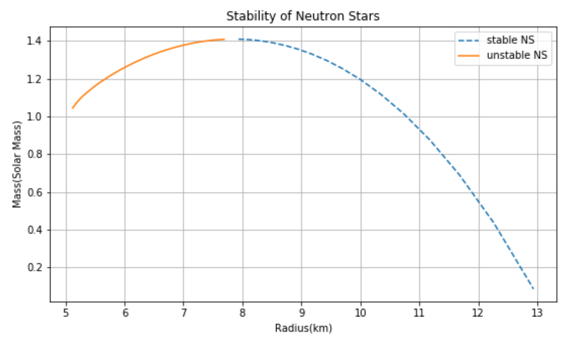

FIG. 11. Stability of NSs

this value by looking at where the derivative zero with a tolerance 10^{-217}. The found value is

$$M_{max} = 1.410421107619103 M_\odot \tag{26}$$

Maximum K value for Neutron Stars

We take $K = 50$ at the beginning of our calculations and find the M_{max}(26). And we know that maximum mass observed for a NS is $2.14 M_\odot$. Therefore there is maximum value for $K = K_{max}$. To find this value we plotted the maximum mass values M_{max} for different K(figure). From the curve it can be seen that around maximum mass $2.14 M_\odot$ we have the K value as

$$K_{max} \approx 239 \tag{27}$$

Since there is no matter outside the star we modify equation 22. Outside the star i.e $r > R$ we have $m(r > R) = M$ also we have $p(r > R) = 0$. Then equation for

[7] MRcurveforNS.py

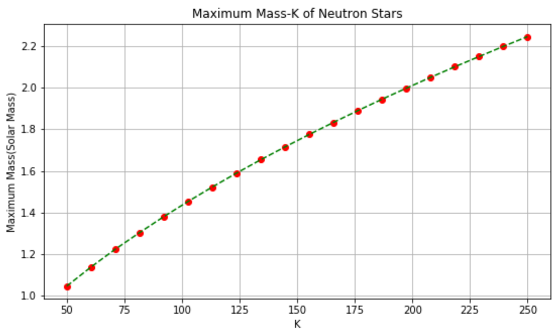

FIG. 12. $M_{max} - K curve$

$\nu(r)$ becomes

$$\nu' = \frac{2M}{r(r-2M)} \tag{28}$$

Using Mathematica we showed that this integration simply gives

$$\mu(r>R) = \ln\left(1-\frac{2M}{r}\right) - \ln\left(1-\frac{2M}{R}\right) + \nu(R) \tag{29}$$

And as we mention we can shift ν by a constant. Since we want $e^{\nu(\infty)} = 1$ i.e. no time dilation for an observer at infinity, we can simply remove the constant terms from Eq29 which yields

$$\bar{\nu}(r>R) = \left(1-\frac{2M}{r}\right) \tag{30}$$

which also satisfies $e^{\bar{\nu}(\infty)} = 1$.

BEYOND EINSTEIN

Wave Equation

A massless real Klein Gordon field obeys the wave equation

$$\Box\phi = 0 \implies \left(-\frac{\partial^2}{\partial t^2}\right) \tag{31}$$

If we assume spherical symmetry and if we are interested in space-time of a NS Modified Wave equation in curved spacetime becomes

$$\begin{aligned} square\phi &= 0 \\ &= -\frac{1}{\sqrt{|g(r)|}}\frac{\partial}{\partial t}\left[e^{-\bar{\nu}}\sqrt{|g(r)|}\frac{\partial\phi}{\partial t}\right] \\ &+ \frac{1}{\sqrt{|g(r)|}}\frac{\partial}{\partial r}\left[\left(1-\frac{2m(r)}{r}\right)\sqrt{|g(r)|}\frac{\partial\phi}{\partial r}\right] \end{aligned} \tag{32}$$

We reduce this equation to first order as follows

$$\partial_t\Phi = \partial_r(f(r)\Pi) \tag{33}$$

$$\partial_t\Pi = \frac{1}{r^2}\partial_r(r^2 f(r)\Phi) \tag{34}$$

where we defined

$$\begin{aligned} \Phi &\equiv \partial_r\phi \\ \Pi &\equiv \frac{1}{f(r)}\partial_t \\ f(r) &\equiv e^{\bar{\nu}(r)/2}\left(1-\frac{2m(r)}{r}\right)^{1/2} \end{aligned} \tag{35}$$

In order to have a smooth physical solutions we need to impose the following boundary conditions

$$\Phi(0) = \partial_r\Pi = 0 \tag{36}$$

$$\lim_{r\to\infty}\Phi \to \frac{1}{r} \tag{37}$$

$$\lim_{r\to\infty}\Pi \to \frac{1}{r} \tag{38}$$

with the following initial conditions

$$\phi(0,r) = 10^{-3}e^{-r^2/2} \tag{39}$$

$$\dot{\phi}(0,r) = 0 \tag{40}$$

We discretize the domain using Leap-Frog Scheme for advection equation.

$$\Phi_j^{n+1} = \Phi_j^{n-1} + \lambda((f_{j+1}\Pi_{j+1}^n) - (f_{j-1}\Pi_{j-1}^n)) \tag{41}$$

$$\Pi_j^{n+1} = \Pi_j^{n-1} + \lambda g_j((\frac{1}{g}f\Phi)_{j+1}^n - (\frac{1}{g}f\Phi)_{j-1}^n) \tag{42}$$

Then just by using Implicit Euler Method we calculated the ϕ

$$\phi_{j+1}^n = \phi_j^n + \Delta r\Phi_{j+1}^n \tag{43}$$

Then, we plotted the wave $\phi(t,r)$ to see how it evolves in time in figure 13. As can be seen, the initial wave changes its amplitude as it moves. It decreases and increases again

In a certain class of alternative theories of gravity named scalar-tensor theories, the wave equation for the scalar is modified as

$$\Box\phi = 4\pi\beta e^{2\beta\phi^2}(\rho - 3p)\phi \tag{44}$$

Therefore we modified our Leap-frog scheme in equation42

$$\Pi_j^{n+1} = \Pi_j^{n-1} + \lambda g_j((\frac{1}{g}f\Phi)_{j+1}^n - (\frac{1}{g}f\Phi)_{j-1}^n) \tag{45}$$

$$+48\pi e^{-12\phi_j^2}(\rho_j - 3p_j)\phi_j^n\Delta t \tag{46}$$

and we got the following result

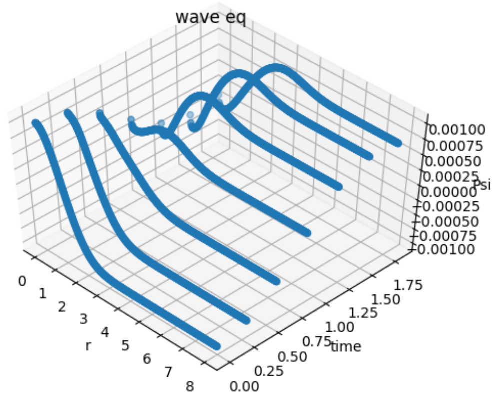

FIG. 13. Solution to wave equation $\phi(t, r)$

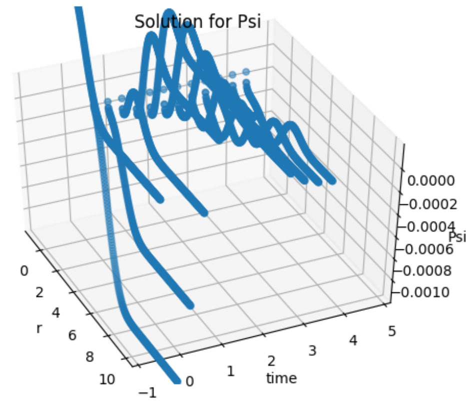

FIG. 14. Solution to wave equation $\phi(t, r)$ with additional term

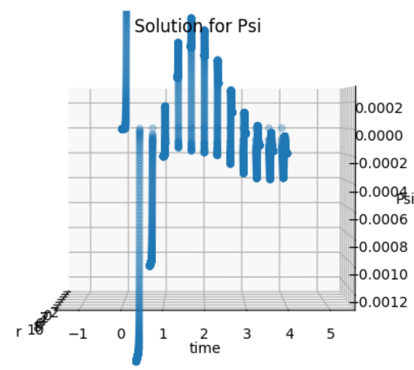

FIG. 15. Side view of the figure 14

As can be seen in plots, after adding the term in Eq.44, $\phi(t, r)$ grows and reaches an almost stable configuration not changing in time.

Acknowledgments: I would like to thank Fethi Mübin Ramazanoğlu who taught me computational physics, helped me pushed my limits, gave me the opportunity to write this paper.